

Steady state Ab-initio Theory of Lasers with Injected Signals

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(Dated: March 13, 2013)

We present an ab-initio treatment of steady-state lasing with injected signals that treats both multimode lasing and spatial hole burning, and describes the transition to injection locking or partial locking in the multimode case. The theory shows that spatial hole burning causes a shift in the frequency of free-running laser modes away from the injection frequency, in contrast to standard approaches.

PACS numbers: 42.55.Px, 42.55.Ah, 42.60.Fc

Laser action in the presence of an injected signal is an extremely important topic both for research in non-linear dynamics and laser physics, and for applications of lasers. Under certain conditions, a laser mode can be locked to the injection frequency, allowing for stabilization and modulation of a “slave” laser based on control by a “master” laser. There has been great interest in such systems as a component in optical phase-locked loops for coherent detection [1] and for stabilization of novel laser systems at desired frequencies for optical communications [2, 3]. However the conditions of stable locking are complicated by a fundamental principle of non-linear dynamics, that a driven, damped system of non-linear oscillators can exhibit complex dynamical behavior and even chaos when three or more independent, time-varying fields are coupled [4]. While the textbook theory of injection locking, described by the Adler equation [5, 6], does not describe such dynamical effects, most of the extensive research on injection locking in the past thirty years has focused on these issues [4]. However there is another aspect of injection locking which has not been treated at all in the literature, the space-dependent non-linear interactions of the injected and self-organized lasing fields. All previous treatments have immediately eliminated the spatial degrees of freedom through approximations which amount to treating the fields as uniform in space [7], hence completely neglecting the effect of spatial hole-burning, present in most real lasers, and making the theoretical treatments qualitative at best. For many of the applications mentioned above, the systems are designed to avoid complex dynamics, and what is needed is a quantitative theory of locking which takes into account the particular features of the laser cavities and gain media involved.

To achieve such a theory, which includes the spatial variation of the injected signal and internal fields, it is critical to have a formulation which describes the open, non-hermitian nature of the system and the spatial variation and interaction of the fields exactly. For multimode lasing without injection such a theory has been developed in the past few years by Tureci, Stone and coworkers [8–12], denoted by the acronym SALT (Steady-state Ab initio Laser Theory). The approach is valid for a wide

range of laser systems, although it has been formulated mainly for those with atomic like gain media. It is a self-consistent theory in which only a few parameters of the gain medium are specified and all physical parameters of the lasing fields are calculated ab initio, e.g. the lasing thresholds, fields and frequencies. Here we show that this theory is generalizable to include arbitrary input fields and an arbitrary number of lasing modes, while remaining computationally tractable, so that injection locking can be quantitatively studied, even for cases with several lasing modes. This new theory, which we term Injection-SALT (I-SALT), assumes a free-running steady-state is achieved, and thus it is meant as an alternative to the Adler theory [5, 6, 13–15] or other steady-state theories [16–18] and not as a theory of the more complex dynamical states of lasers that can arise due to injected signals. We find below a novel effect of frequency repulsion from the injected signal which is not predicted by any previous theory. I-SALT is shown to agree quantitatively with full time-dependent numerical solutions of the laser equations for a simple one-dimensional cavity with an injected signal, but because of its much greater computational efficiency [19], it will be feasible to apply it to realistic laser structures which are impractical to study using such direct simulation.

SALT was formulated to determine directly the steady-state of the laser rate equations for an N-level atomic gain medium coupled to Maxwell’s equations within an arbitrary cavity specified by its passive dielectric function, $\epsilon_c(x)$, subject to a spatially-varying pump, $D_0(x) = d_0 F(x)$, without performing time-integration to steady-state. It was shown that if a multi-mode steady-state exists, which requires stationary level populations, then all of the properties of the laser field are determined by a set of coupled time-independent wave equations with a saturating non-linearity that self-consistently couples the lasing modes. These equations are non-Hermitian at the linear level, and are subject to purely outgoing boundary conditions. They are solved efficiently by introducing a specific self-orthogonal basis set of threshold constant flux (TCF) states in which to expand the solutions, and then iteratively solving the resulting non-linear matrix equation for both fields and frequencies. We now sketch

how this approach can be generalized to yield I-SALT.

As in SALT, we assume the existence of a steady-state with stationary level populations; then the positive frequency components of the electric field inside the cavity for a given pump value, d_0 , take the form

$$E^+(x, t) = \sum_{\mu}^{N_L} \Psi_{\mu}(x) e^{-i\omega_{\mu}t} + \sum_{\alpha}^{N_A} \Psi_{\alpha}(x) e^{-i\omega_{\alpha}t}, \quad (1)$$

where the N_L lasing modes, $\Psi_{\mu}(x)$, have unknown spatial variation and unknown frequencies, ω_{μ} , and there are N_A amplified signals injected into the cavity, at given frequencies, ω_{α} , and incoming amplitudes, B_{α} , but with unknown amplitude and spatial variation, $\Psi_{\alpha}(x)$, within the cavity. All of the unknown quantities will be determined from the resulting I-SALT equations and their boundary conditions self-consistently.

We have shown elsewhere that within the stationary population approximation (SPA) any N-level atomic laser system can be mapped onto an effective two-level model [19], so here we consider an effective two-level Maxwell-Bloch model in which only the pump, d_0 , and inversion relaxation rate appear, and the latter is absorbed into the units of the electric field [20]. Inserting the multi-periodic ansatz (1) into the coupled equations for the field, polarization and atomic population inversion, and applying the SPA allows elimination of the latter two variables, leading to $N_L + N_A$ coupled non-linear wave equations, which can be written as three-dimensional vectorial equations, but which here we will only consider in their scalar form, appropriate for the one-dimensional slab geometry studied below:

$$\left[\nabla^2 + \left(\varepsilon_c(x) + \frac{\gamma_{\perp} D(x)}{\omega_{\sigma} - \omega_a + i\gamma_{\perp}} \right) k_{\sigma}^2 \right] \Psi_{\sigma}(x) = 0 \quad (2)$$

$$D(x) = \frac{d_0 F(x)}{1 + \sum_{\mu}^{N_L} \Gamma_{\mu} |\Psi_{\mu}(x)|^2 + \sum_{\alpha}^{N_A} \Gamma_{\alpha} |\Psi_{\alpha}(x)|^2}, \quad (3)$$

where $\Gamma_{\nu} \equiv \gamma_{\perp}^2 / [(\omega_{\nu} - \omega_a)^2 + \gamma_{\perp}^2]$ is the gain curve, γ_{\perp} is its linewidth, ω_a is the atomic frequency of the lasing transition, $k_{\sigma} = \omega_{\sigma}/c$ is the wavevector, and σ is an index running over both lasing states, μ , and amplified states, α . The equations for $\Psi_{\mu}(x)$ are to be solved with purely outgoing boundary conditions, while those for $\Psi_{\alpha}(x)$ are to be solved with the boundary condition of fixed input amplitude at ω_a .

We will solve these coupled equations by non-linear iteration after expanding the solutions in a non-Hermitian basis set with the appropriate boundary conditions. For the lasing modes, Ψ_{μ} , this set is the same TCF states used in SALT, which satisfy,

$$[\nabla^2 + (\varepsilon_c(x) + \eta_n F(x)) k^2] u_n(x; \omega) = 0 \quad (4)$$

$$\partial_x u_n(x; \omega)|_{x=L} = ik u_n(L; \omega), \quad (5)$$

where we refer to $u_n(x; \omega)$ and $\eta_n(\omega)$ as the TCF eigenvectors and eigenvalues respectively and for simplicity we have written the outgoing boundary condition for a one-sided cavity with a perfect mirror at the origin. Note that $\{\eta_n(\omega)\}$ is generically complex and can be thought of as the set of values of the gain medium susceptibility which lead to lasing, i.e. for which a solution for purely outgoing real wavevector exists. Thus the lasing thresholds in the absence of input signals are obtained by varying ω in the TCF equation until a frequency, ω_{μ} , is found at which $\eta_n(\omega_{\mu}) = d_0 \gamma_{\perp} / (\omega_{\mu} - \omega_a + i\gamma_{\perp})$. The $\{u_n(x; \omega_{\mu})\}$ then form an efficient basis set for finding the *non-linear* solutions above threshold because at the first lasing threshold, the lasing mode is just a single TCF state, and above threshold only a small number of TCFs are needed to converge to the non-linear solution of the SALT equations.

The amplified modes, Ψ_{α} , however must be treated differently from the lasing modes since they have a fixed incoming signal amplitude and fixed frequency. To represent these modes we require terms with an incoming component in addition to the outgoing TCF expansion terms, which we do conveniently by solving the same TCF equation inside the cavity with a purely incoming boundary condition,

$$\Psi_{\alpha}(x) = \sum_n a_n^{(\alpha)} u_n(x; \omega_{\alpha}) + \sum_m b_m^{(\alpha)} v_m(x; \omega_{\alpha}), \quad (6)$$

where the states $v_m(x; \omega)$ and associated eigenvalues β_m are given by

$$[\nabla^2 + (\varepsilon_c(x) + \beta_m F(x)) k^2] v_m(x; \omega) = 0 \quad (7)$$

$$\partial_x v_m(x; \omega)|_{x=L} = -ik v_m(L; \omega), \quad (8)$$

and thus represent states that are purely incoming. The incoming and outgoing TCF states are not power orthogonal, but they do satisfy a self-orthogonality condition between themselves [21]. Either the incoming or outgoing TCF states represent a complete basis for fields within the cavity at ω_{α} , but the incoming terms are needed to represent the input boundary condition. Because they are purely incoming, they do not contribute directly to the emitted fields, but they correctly represent the full spatial hole-burning and gain competition effects of the amplified input.

For amplified modes, we can easily write the incoming boundary condition for a one-sided slab cavity of length L as

$$B_{\alpha} e^{-ik_{\alpha} L} = \sum_m b_m^{(\alpha)} v_m(L; \omega_{\alpha}), \quad (9)$$

where B_{α} is the given incoming field amplitude at frequency ω_{α} . This single equation vastly underdetermines the coefficients $b_m^{(\alpha)}$ in the sum, so that the choice is based on convenience. This freedom arises from the over-completeness of using both $\{u_n\}$ and $\{v_m\}$ to represent

the internal fields. Hence the coefficients $a_n^{(\alpha)}$ depend strongly on the choice of the $b_m^{(\alpha)}$. A natural choice is to take only a single term, $v_0(x; \omega_\alpha)$, which corresponds to the dominant component of the outgoing TCF state for the nearest lasing mode. This is allowed for a cavity with a single input channel, as in the one-sided slab geometry we are considering here; in general one needs a minimum of M independent incoming states to represent an arbitrary input for an M -channel cavity, and these can be chosen again to be similar in character to the nearest lasing mode in order to optimize the calculation.

Once a representation of the input field is chosen, one can insert the expansions (6) for the amplified modes and the analogous expansion involving only outgoing TCFs for the lasing modes into the fundamental Eqs. (2) and (3) and use the self-orthogonality relations of the u_n, v_m to find coupled non-linear matrix equations for the coefficient $a_n^{(\mu)}, a_n^{(\alpha)}, b_m^{(\alpha)}$ which determine their solutions. For the lasing modes, Ψ_μ , one finds

$$\eta_l a_l^{(\mu)} = \sum_n T_{ln}^{(\mu)} a_n^{(\mu)} \quad (10)$$

$$T_{ln}^{(\mu)} = \gamma_\mu d_0 \int_C dx \frac{F(x) u_l(x; \omega_\mu) u_n(x; \omega_\mu)}{1 + \sum_\nu^{(N_L + N_A)} \Gamma_\nu |\Psi_\nu(x)|^2}, \quad (11)$$

where $\gamma_\mu = \gamma_\perp / (\omega_\mu - \omega_a + i\gamma_\perp)$. This is identical to the lasing equations of SALT except for the presence of the amplified mode intensities in the non-linear hole-burning denominator.

In a similar manner the coupled equations for the amplified modes can be determined, and they take the form [22]:

$$\eta_l a_l^{(\alpha)} = \sum_n T_{ln}^{(\alpha)} a_n^{(\alpha)} + \sum_m \left(W_{lm}^{(\alpha)} + V_{lm}^{(\alpha)} \right) b_m^{(\alpha)} \quad (12)$$

$$V_{lm}^{(\alpha)} = \frac{2i}{k_\alpha} \frac{\beta_m}{\beta_m - \eta_l} u_l(L; \omega_\alpha) v_m(L; \omega_\alpha). \quad (13)$$

The operator $W_{lm}^{(\alpha)}$ in (12) takes precisely the same form as $T_{ln}^{(\mu)}$ except that $u_n(x) \rightarrow v_m(x)$.

Together Eqs. (10-13) define I-SALT. In the regime in which the SPA holds, they provide essentially exact solutions of the full coupled wave equations for amplification and injection locking, which has not been achieved previously, to our knowledge. The method is *ab initio*, as in SALT, with no prior assumptions about the number, spatial form or frequencies of the lasing modes. The physical picture of injection locking in this regime is one of cross-saturation, and not synchronization of non-linear oscillators, as it is in the Adler theory. Lasing modes correspond to poles of the non-linear scattering matrix on the real axis; amplified inputs are simply additional scattered waves which also deplete the gain. If the input signal becomes too strong, and is sufficiently near in frequency to the lasing mode, then the lasing mode has insufficient gain and falls below threshold (the pole falls

below the real axis [22]), leaving only the amplified signal output. The output is “locked” to the input frequency, but not by pulling the lasing mode over to ω_α , but rather by turning it off. This scenario is appropriate when all the relevant frequencies or rates in the problem are large compared to the relaxation rate of the gain, so that coherent interactions of the fields are averaged away. This is a common regime for solid state lasers with atomic like gain media and essentially the same conditions hold in the stable locking regime of semiconductor lasers.

To test the results of I-SALT, in Fig. 1 we compare them to the exact numerical solutions based on FDTD for a simple slab cavity with an injected signal (schematic) under the standard conditions of a single lasing mode in the absence of the signal. Locking of the output signal is shown in the top panel based on I-SALT, in excellent agreement with FDTD data, with no adjustable parameters. The value of the output signal leading to locking found by I-SALT is approximately the same, but larger than predicted by the Adler theory. In Adler theory one finds that the energy output from the gain medium at the locking threshold and in the free-running state with no input are exactly equal, and typically this is interpreted as due to “energy conservation”. But this is not quantitatively correct in a more complete theory, because, due to spatial hole-burning, it is possible for two output fields to extract more energy from the gain medium in steady-state than a single one, and indeed we find that the amplified output at locking in Fig. 1 is $\sim 18\%$ larger than the original free-running output.

Moreover, the frequency behavior of the lasing mode prior to locking is qualitatively opposite to the Adler theory; the frequency is repelled from the input frequency instead of being attracted to it. The repulsion here is distinct from that observed by Murakami et al. [15], arising from the Henry α -factor, which is negligible in atomic gain media. This frequency repulsion can be explained by the combined effects of mode competition and spatial hole-burning. As the incident signal is imposed and depletes the gain, the standing wave of the laser field shifts away from the frequency/wavelength of the incident standing wave in order to better extract energy from the regions of the cavity where the gain is not being saturated by the incident signal. To confirm this interpretation we replace the space dependent gain saturation denominator with its constant spatial average in the I-SALT equations and find no substantial frequency shift.

As noted, the Adler theory describes locking driven by phase synchronization of the input and free-running fields. Since the threshold input intensity for locking decreases to zero as the input frequency approaches the free-running frequency, the threshold condition can be expressed as a “locking range”, the frequency range $\Delta\omega$, over which the laser is locked for a given input intensity. In the Adler theory one finds $\Delta\omega = \gamma_c \sqrt{|B_\alpha|^2 / I_{\mu,0}}$, where γ_c is the cavity decay rate, $|B_\alpha|^2$ is the intensity of

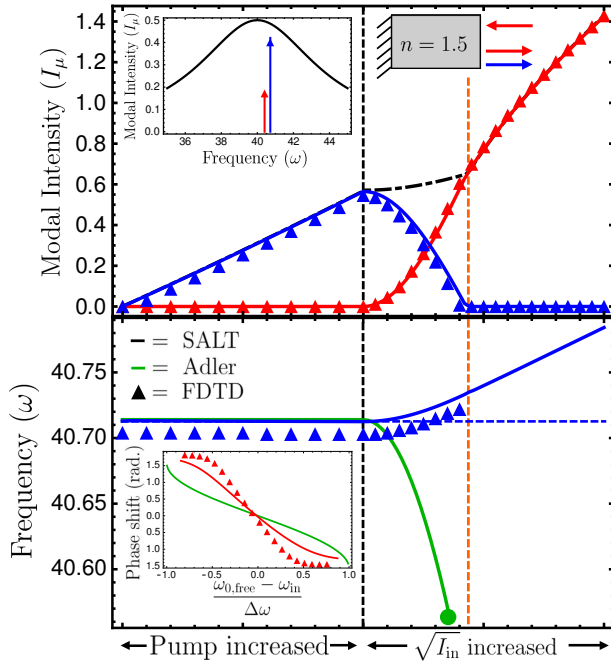


FIG. 1. (Top panel) Injection locking transition as described in the text. Solid lines are output intensities calculated from I-SALT; blue is lasing output, red is amplified output at signal frequency, dot-dashed black is total output. Triangles are the same quantities from FDTD for the same dielectric slab laser with $\omega_a = 40$, $\gamma_\perp = 4$ (schematic inset). To the left of the black dashed vertical line the laser is free-running with no input signal and output at $\omega_{1,free} = 40.714$; to the right an input signal at $\omega_{in} = 40.4$ is applied and increased (see top inset), eventually driving the lasing signal to zero at the locking transition (orange dashed line). (Bottom panel) Frequency variation of the first lasing mode: blue line is from I-SALT and blue triangles from FDTD. The green line shows the prediction of the Adler equations, the green circle showing where locking is predicted. Again, the orange dashed line shows locking threshold from I-SALT, frequencies beyond this point are taken as the real part of the location of the pole of the scattering matrix. As the locking transition is approached the lasing frequency moves away from its free-running value, due to spatial hole-burning in contrast to the behavior expected from the Adler equation [5]. Blue dashed lines show I-SALT calculation with spatially averaged hole-burning. The inset shows a plot of the phase shift between input and output signals of an injection locked dielectric slab cavity at a fixed input intensity. I-SALT (red) and FDTD simulations (red triangles) are seen to have close agreement and a smaller locking range than the Adler prediction (green). Values quoted are given in units of c/L .

the input signal, and $I_{\mu,0}$ is the intensity of the free running lasing signal in the absence of the input [5]. Within this locking range there is a fixed phase relationship between the input signal and the locked output which varies as $\Delta\phi = \arcsin[\omega_{\mu,0} - \omega_\alpha/\Delta\omega]$.

The same quantity can be calculated in I-SALT and is compared to the Adler prediction for the same slab cavity in the inset of Fig. 1. The phase shift variation found

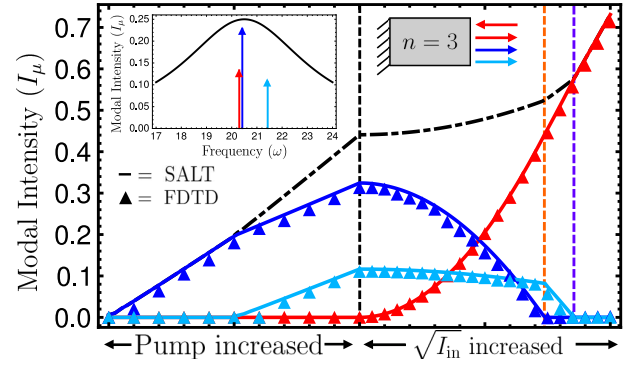


FIG. 2. Partial locking transition as described in the text using a similar pumping and input ramping scheme as Fig. 1. Solid lines are output intensities calculated from I-SALT; blue and cyan are lasing output, red is amplified output at signal frequency, $\omega = 20.3$, dot-dashed black is total output. Triangles are the same quantities from FDTD for the same dielectric slab laser with $\omega_a = 20.5$, $\gamma_\perp = 3$ (schematic inset). First the nearby lasing mode locks to the injected signal (orange line), then the distant lasing mode locks (purple line). Values quoted are given in units of c/L .

from I-SALT is completely different from the Adler prediction and in good agreement with FDTD. Note that I-SALT also finds a slightly smaller locking range than Adler theory, a reflection of the fact that the input intensity for locking is larger than that predicted by Adler theory.

Beyond yielding a better treatment of the locking transition of a single mode laser, as seen from the generality of Eqs. (10-13), I-SALT is able to treat simultaneously multimode lasing with multiple inputs. An example of this with two mode lasing and a single input signal is shown in Fig. 2. Here a similar slab laser is pumped above its second lasing threshold and a signal is injected closer to the frequency of the first lasing mode. Because of its stronger interaction with the first mode (blue), the signal is able to lock that mode, while the second mode (cyan) remains active at a similar frequency to its free-running value, though shifted away from the injected frequency in the same manner as described before. Thus we have a “partially-locked” state of a type not treated in any previous theories. Again, excellent agreement is found between I-SALT and FDTD, but with substantially less computational effort, I-SALT takes an hour of computational time on modern CPUs to generate the curves seen in Fig. 2 while FDTD requires 168 days. In general, exact FDTD studies of multimode lasing are computationally very demanding and could not be performed in more realistic structures, whereas SALT has been shown to be computationally tractable in complex two-dimensional structures such as photonic crystals [23] and random lasers [10], and 3D vectorial codes are currently under development and look promising. Since I-SALT is essentially of the same degree of computational complexity as SALT, it can be used as a tool to study

frequency control of both single and multimode lasing, and opens up new possibilities for investigating injection locking systems.

We thank Stephen O'Brien, Kerry Vahala, Arthur Goestchy, and Hui Cao for helpful discussions. We thank the Yale High Performance Computing Center. This work was supported by NSF grant No. DMR-0908437.

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